DAMPED VIBRATION ABSORBER FORCE RATIOS

BY

Robert E. Murray

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ABSTRACT

Torsional vibration dampers are commonly used in diesel engines to reduce the effects of unwanted vibrational disturbances. At present some of the most widely used dampers are those of the Houde type as discussed on page 211 of Reference 1. These dampers, however, fall considerably short in practice of the effectiveness that is theoretically possible.

On February 15, 1966 patent number 3,234,817 was awarded to S. O. Williamson for a Tuned Torsional Vibration Damper. The analysis of this damper requires characteristics which heretofore have not been published. The object of this thesis work then, is to provide these necessary characteristics.

The only motion described in the literature is that of the main mass, x_1, as given on page 96 of Reference 1. The motion of the absorber mass, x_2, and the relative motion between the main mass and the absorber mass, x_1-x_2, is now supplied by equations (6) and (13) respectively on page 13 of this thesis. A sample plot of these three motions is presented on page 17 for specific conditions.

Also unpublished are the following three ratios: 1.) the Spring Force to the Damping Force; 2.) the Spring Force to the

Inertia Force; and, 3.) the Inertia Force to the Damping Force. These ratios are now supplied by equations (27), (28), and (29) respectively on page 15 of this report. Sample curves are presented for a special case on pages 18,19 , and 20.

These results, that is equations (6), (13), (27), (28), and (29) , are required for a rational calculation of the Williamson type damper.

Thesis Supervisor: J.P. Den Hartog

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

Symbol	Meaning
c	Damping Coefficient
$c_c = 2m\Omega_n$	Critical damping
$f = \omega_a/\Omega_n$	Frequency ratio (natural frequencies)
$g = \omega/\Omega_n$	Forced frequency ratio
K	Spring constant (supporting structure)
k	Spring constant (vibration absorber)
M	Mass (main system)
m	Mass (vibration absorber)
$\mu = m/M$	Mass ratio (absorber mass/main mass)
x_1	Motion of main mass
x_2	Motion of absorber mass
$x_1 - x_2$	Relative motion between main mass and absorber mass
$x_{st} = P_o/K$	Static deflection
ω	Forcing frequency
$\omega_a^2 = k/m$	Natural frequency of absorber
$\Omega_n^2 = K/M$	Natural frequency of main system

INTRODUCTION

A damped dynamic vibration absorber may be represented schematically as shown in Fig. 1. The primary system consists of the main mass, M, and the supporting structure, K. Attached to the main mass is the vibration absorber consisting of springs, k, a dashpot, c, and a mass, m. The disturbing force is assumed to be sinusoidal with a maximum amplitude of P_0. This force acts directly on the main mass.

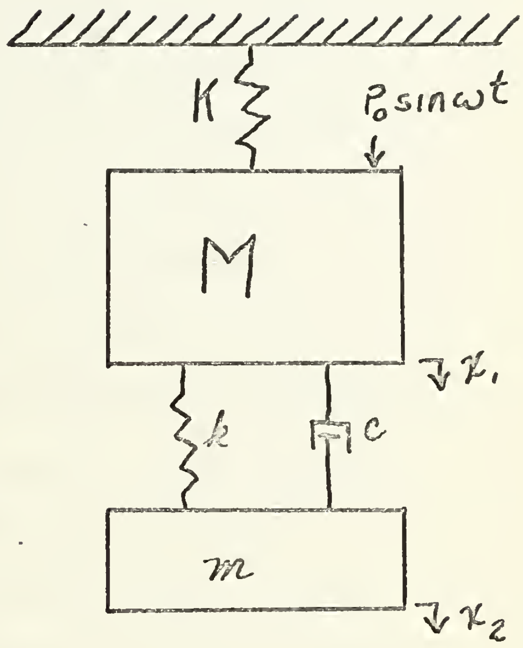

Fig. 1. DAMPED VIBRATION ABSORBER

Whenever obnoxious vibrations are present in the main system a vibration absorber is utilized to eliminate or to at least minimize the effects of these disturbances. By judicious selection of the components of the absorber, that is, k, c, and m, the effect of the absorber may be maximized which in turn minimizes the disturbance to the main system. This process is called tuning and is discussed in detail on pages 97 through 103 in Reference 1.

At present solutions exist for the optimum damping and optimum tuning of the absorber based on the assumption that the springs are linear and the damping coefficient is a constant. This data is sufficient to optimize the vibration absorber for a linear system. These solutions are presented in Reference 1. However,

these solutions require springs that must withstand extensions three or four times as large as the motions of the main system. There is difficulty in designing springs that will withstand the cyclic fatigue and displacements and still be of a size commensurate with the space allocated for these springs.

With the feasibility of using visco-elastic materials in vibration absorbers as discussed in Reference (2) and with the Williamson Tuned Torsional Vibration Damper, the present information is not adequate to optimize the design of absorbers using this visco-elastic material. The purpose of this thesis is, therefore, to provide the necessary characteristics to facilitate damper design. Equations describing the motion of the absorber mass and the relative motion between the main and absorber masses will be derived. Three ratios will also be presented. They are: 1.) the Spring Force to the Damping Force; 2.) the Spring Force to the Inertia Force; and, 3.) the Inertia Force to the Damping Force. With these additional characteristics, analysis and design of dampers will be facilitated.

PROCEDURE

The first step is to calculate the three motions:

1. x_1, the motion of the main mass.
2. x_2, the motion of the absorber mass.
3. x_1-x_2, the relative motion between the main mass and the absorber mass.

Begining with Newton's Law applied to the main mass, M, and the absorber mass, m:

$$
(1) \quad \begin{aligned} M\ddot{x}_1 + Kx_1 + k(x_1 - x_2) + c(\dot{x}_1 - \dot{x}_2) &= P_0 \sin\omega t \\ m\ddot{x}_2 + k(x_2 - x_1) + c(\dot{x}_2 - \dot{x}_1) &= 0 \end{aligned}
$$

Because in most cases transient vibrations vanish rapidly with respect to the time the particular machine is in operation only solutions for forced vibrations will be determined. Therefore, x_1 and x_2 may be represented as vectors rotating with angular velocity ω. Equations (1) then become after some rearrangement:

$$
(2) \quad \left[-M\omega^2 + K + k + j\omega c\right]x_1 - \left[k + j\omega c\right]x_2 = P_0
$$

$$
(3) \quad -\left[k + j\omega c\right]x_1 + \left[-m\omega^2 + k + j\omega c\right]x_2 = 0
$$

where x_1 and x_2 are unknown complex numbers and j, with the usual convention, $j = \sqrt{-1}$.

Reference (1) presents a solution for x_1 and hence will not be repeated.

The motion of the absorber, x_2, is calculated in much the same manner as was x_1 and therefore will be presented very briefly. From equation (3)

$$(4) \qquad x_1 = \frac{[-m\omega^2 + k + j\omega c]\,x_2}{[k + j\omega c]}$$

Substituting this result into equation (2) yields:

$$(5) \quad \frac{[-M\omega^2 + K + k + j\omega c][-m\omega^2 + k + j\omega c]\,x_2}{[k + j\omega c]} - [k + j\omega c]\,x_2 = P_0$$

which reduces to:

$$(6) \quad x_2 = \frac{P_0[k + j\omega c]}{[(-M\omega^2 + K)(-m\omega^2 + k) - m\omega^2 k] + j\omega c[-M\omega^2 + K - m\omega^2]}$$

From Reference (1)

$$(7) \quad x_1 = \frac{P_0[(k - m\omega^2) + j\omega c]}{[(-M\omega^2 + K)(-m\omega^2 + k) - m\omega^2 k] + j\omega c[-M\omega^2 + K - m\omega^2]}$$

To calculate the relative motion between the main mass and the absorber mass, $x_1 - x_2$, we first notice that x_1 and x_2 are in the form:

$$(8) \qquad \frac{x_1}{P_0} = \frac{A + jB}{C + jD}$$

and

$$(9) \qquad \frac{x_2}{P_0} = \frac{E + jB}{C + jD}$$

By multiplying (8) and (9) by their respective complex con-

jugates the following is obtained.

$$(10)\quad \frac{x_1}{P_0} = \frac{A+jB}{C+jD}\cdot\frac{C-jD}{C-jD} = \frac{(AC+BD) + j(BC-AD)}{C^2+D^2}$$

and

$$(11)\quad \frac{x_2}{P_0} = \frac{E+jB}{C+jD}\cdot\frac{C-jD}{C-jD} = \frac{(EC+BD) + j(BC-ED)}{C^2+D^2}$$

Because x_1 and x_2 are represented vectorially x_1-x_2 may be obtained by normal vector subtraction.

$$\frac{x_1-x_2}{P_0} = \frac{(AC+BD)-(EC+BD) + j[(BC-AD)-(BC-ED)]}{C^2+D^2}$$

$$= \frac{C(A-E) - jD(A-E)}{C^2+D^2} = (A-E)\left[\frac{C-jD}{C^2+D^2}\right]$$

$$(12)\quad \frac{x_1-x_2}{P_0} = \frac{A-E}{C+jD}$$

Hence by relation (12)

$$(13)\quad \frac{x_1-x_2}{P_0} = \frac{-m\omega^2}{[(-M\omega^2+K)(-m\omega^2+k)-m\omega^2 k] + j\omega c[-M\omega^2+K-m\omega^2]}$$

In summary, the three equations describing the motions of the damped dynamic vibration absorber subjected to a disturbing force $P_0 \sin\omega t$ are:

$$(7)\quad \frac{x_1}{P_0} = \frac{(k-m\omega^2) + j\omega c}{[(-M\omega^2+K)(-m\omega^2+k)-m\omega^2 k] + j\omega c[-M\omega^2+K-m\omega^2]}$$

(6) $$\frac{X_2}{P_o} = \frac{k + j\omega c}{\left[(-M\omega^2+K)(-m\omega^2+k) - m\omega^2 k\right] + j\omega c\left[-M\omega^2 + K - m\omega^2\right]}$$

(13) $$\frac{X_1 - X_2}{P_o} = \frac{-m\omega^2}{\left[(-M\omega^2+K)(-m\omega^2+k) - m\omega^2 k\right] + j\omega c\left[-M\omega^2 + K - m\omega^2\right]}$$

To find the amplitude of each of the three motions we also notice that each motion may be represented by some form of:

$$X = \frac{A + jB}{C + jD}$$

It follows directly, Reference (1), then that the magnitude of X is:

(14) $$X = \sqrt{\frac{A^2 + B^2}{C^2 + D^2}}$$

Using this result,(14), the magnitudes of each vector is:

(15) $$\frac{X_1}{P_o} = \sqrt{\frac{(k - m\omega^2)^2 + \omega^2 c^2}{\left[(-M\omega^2+K)(-m\omega^2+k) - m\omega^2 k\right]^2 + \omega^2 c^2\left[-M\omega^2 + K - m\omega^2\right]^2}}$$

(16) $$\frac{X_2}{P_o} = \sqrt{\frac{k^2 + \omega^2 c^2}{\left[(-M\omega^2+K)(-m\omega^2+k) - m\omega^2 k\right]^2 + \omega^2 c^2\left[-M\omega^2 + K - m\omega^2\right]^2}}$$

(17) $$\frac{X_1 - X_2}{P_o} = \sqrt{\frac{m^2\omega^4}{\left[(-M\omega^2+K)(-m\omega^2+k) - m\omega^2 k\right]^2 + \omega^2 c^2\left[-M\omega^2 + K - m\omega^2\right]^2}}$$

Non-dimensionalizing each of the above three magnitudes the following is obtained.

(18) $$\frac{x_1}{x_{st}} = \sqrt{\frac{(2\tfrac{c}{c_c}g)^2 + (g^2-f^2)^2}{(2\tfrac{c}{c_c}g)^2(g^2-1+\mu g^2)^2 + [\mu g^2 f^2 - (g^2-1)(g^2-f^2)]^2}}$$

(19) $$\frac{x_2}{x_{st}} = \sqrt{\frac{f^4 + (2\tfrac{c}{c_c}g)^2}{(2\tfrac{c}{c_c}g)^2(g^2-1+\mu g^2)^2 + [\mu g^2 f^2 - (g^2-1)(g^2-f^2)]^2}}$$

(20) $$\frac{x_1-x_2}{x_{st}} = \sqrt{\frac{g^4}{(2\tfrac{c}{c_c}g)^2(g^2-1+\mu g^2)^2 + [\mu g^2 f^2 - (g^2-1)(g^2-f^2)]^2}}$$

The second step is to calculate the three force ratios. They are:

1. Spring Force / Damping Force
2. Spring Force / Inertia Force
3. Inertia Force / Damping Force

With the following three relationships:

(21) Spring Force = $k(x_1 - x_2)$

(22) Damping Force = $c\omega(x_1 - x_2)$

(23) Inertia Force = $m\omega^2 x_2$

the three ratios follow directly.

(24) Spring Force/Damping Force = $\dfrac{k(x_1-x_2)}{c\omega(x_1-x_2)} = \dfrac{k}{c\omega}$

(25) Spring Force/Inertia Force = $\dfrac{k(x_1-x_2)}{m\omega^2 x_2} = \dfrac{k\frac{(x_1-x_2)}{x_{st}}}{m\omega^2(x_2/x_{st})}$

(26) Inertia Force/Damping Force = $\dfrac{m\omega^2 x_2}{c\omega(x_1-x_2)} = \dfrac{m\omega(x_2/x_{st})}{c\left(\frac{x_1-x_2}{x_{st}}\right)}$

Non-dimensionalizing and reducing each of these ratios yields:

(27) Spring Force/Damping Force = $\dfrac{f^2}{2\,{}^c\!/\!_{c_c}\,g}$

(28) Spring Force/Inertia Force = $\dfrac{f^2}{\sqrt{f^4 + (2\,{}^c\!/\!_{c_c}\,g)^2}}$

(29) Inertia Force/Damping Force = $\sqrt{\dfrac{f^4 + (2\,{}^c\!/\!_{c_c}\,g)^2}{2\,{}^c\!/\!_{c_c}\,g}}$

By using the optimum tuning, (f), and the average optimum damping, (c/c_c), as derived on pages 100 and 103 of Reference (1):

$$f = \frac{1}{1+\mu} \quad ; \quad {}^c\!/\!_{c_c} = \sqrt{\frac{3\mu}{8(1+\mu)^3}}$$

the three ratios become:

(30) Spring Force/Damping Force = $\sqrt{\dfrac{2}{3\mu(1+\mu)g^2}}$

(31) Spring Force/Inertia Force = $\dfrac{1}{\sqrt{\dfrac{3\mu(1+\mu)g^2}{2} + 1}}$

(32) Inertia Force/Damping Force = $\sqrt{\dfrac{2}{3\mu(1+\mu)g^2} + 1}$

RESULTS

A. Figure 2 shows the motion of the main mass, x_1, the absorber mass, x_2, and the relative motion between the two, x_1-x_2, versus the forced frequency ratio. Optimum tuning and optimum damping was used for a given mass ratio, $\mu = 0.25$.

B. Figure 3 is the ratio of the Spring Force to the Damping Force versus the reciprocal of the mass ratio. Optimum tuning, the average optimum damping, and the resonant forced frequency ratios were used in this and the two succeeding figures.

C. Figure 4 is the ratio of the Spring Force to the Inertia Force versus the reciprocal of the mass ratio.

D. Figure 5 is the ratio of the Inertia Force to the Damping Force versus the reciprocal of the mass ratio.

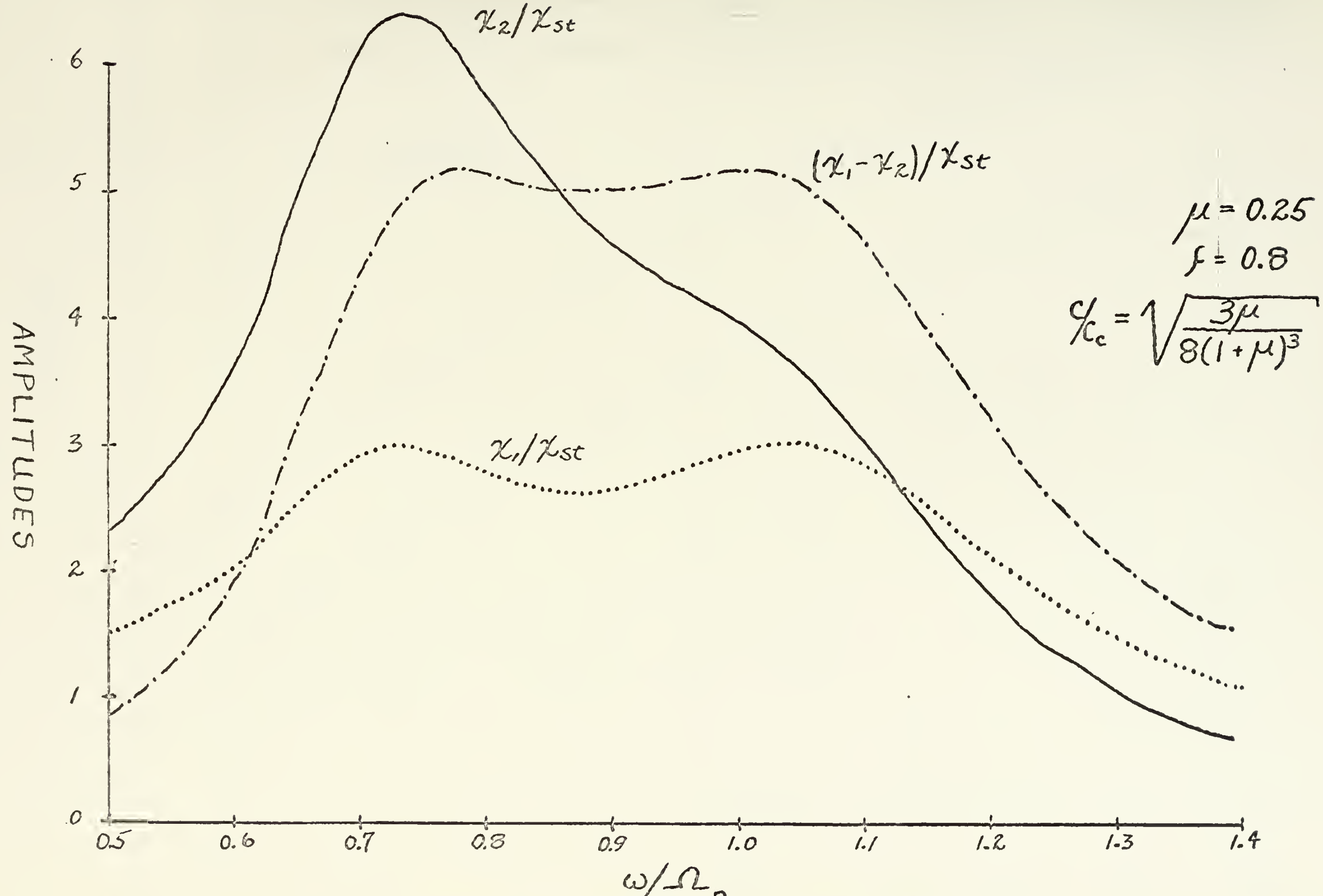

FIGURE 2. DISPLACEMENTS VERSUS FORCED FREQUENCY RATIO.

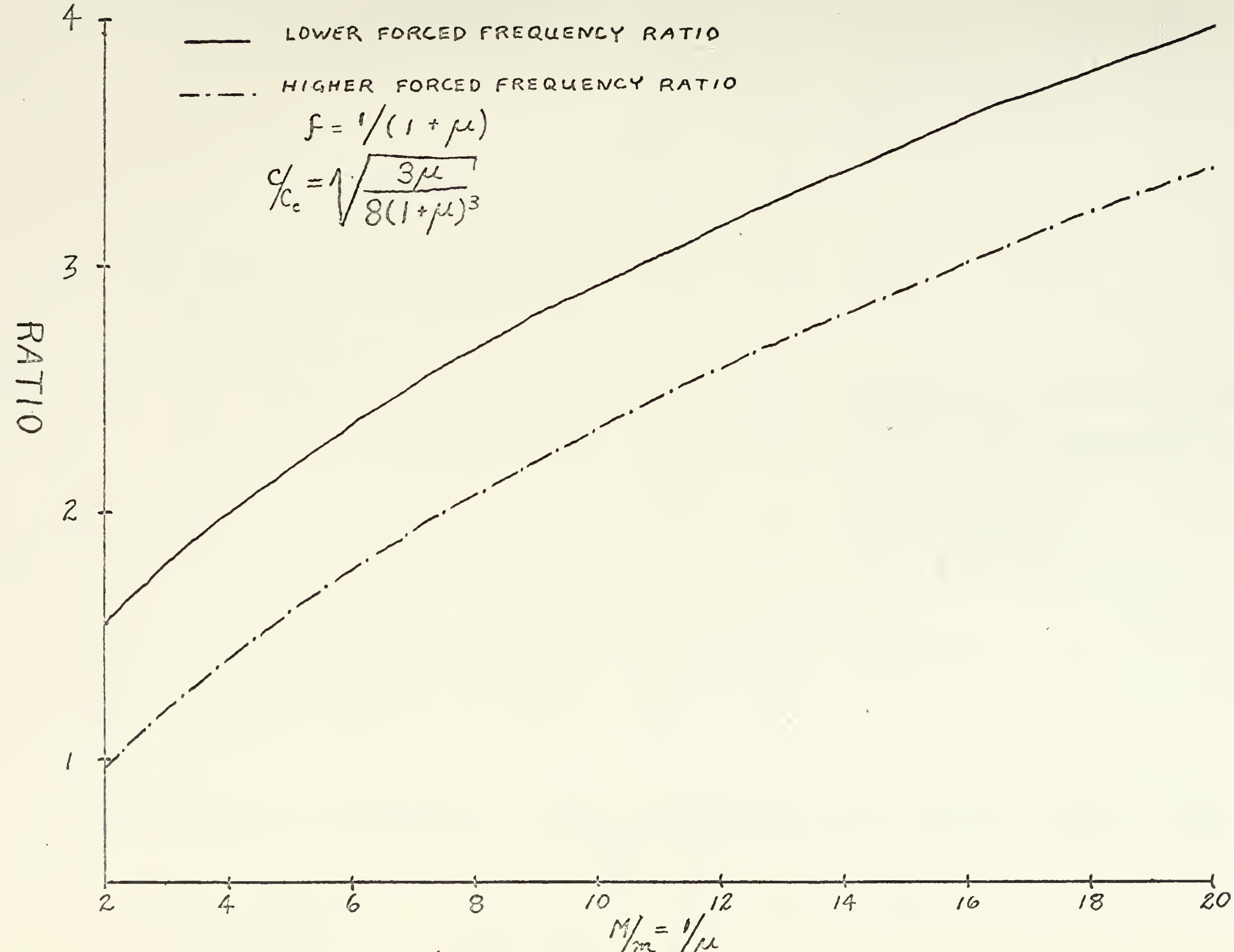

FIGURE 3. SPRING FORCE/DAMPING FORCE VERSUS RECIPROCAL OF MASS RATIO.

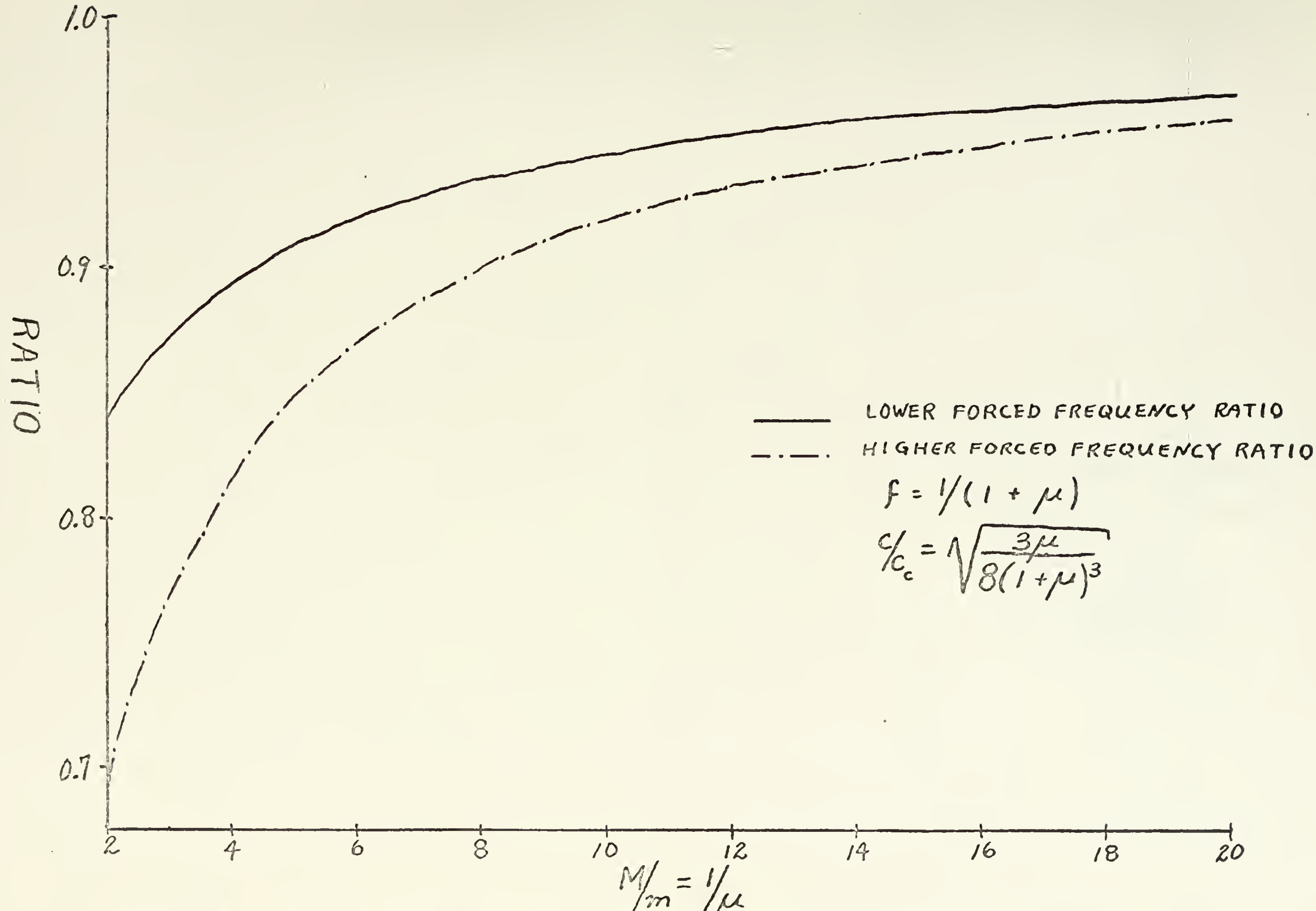

FIGURE 4. SPRING FORCE/INERTIA FORCE VERSUS RECIPROCAL OF MASS RATIO.

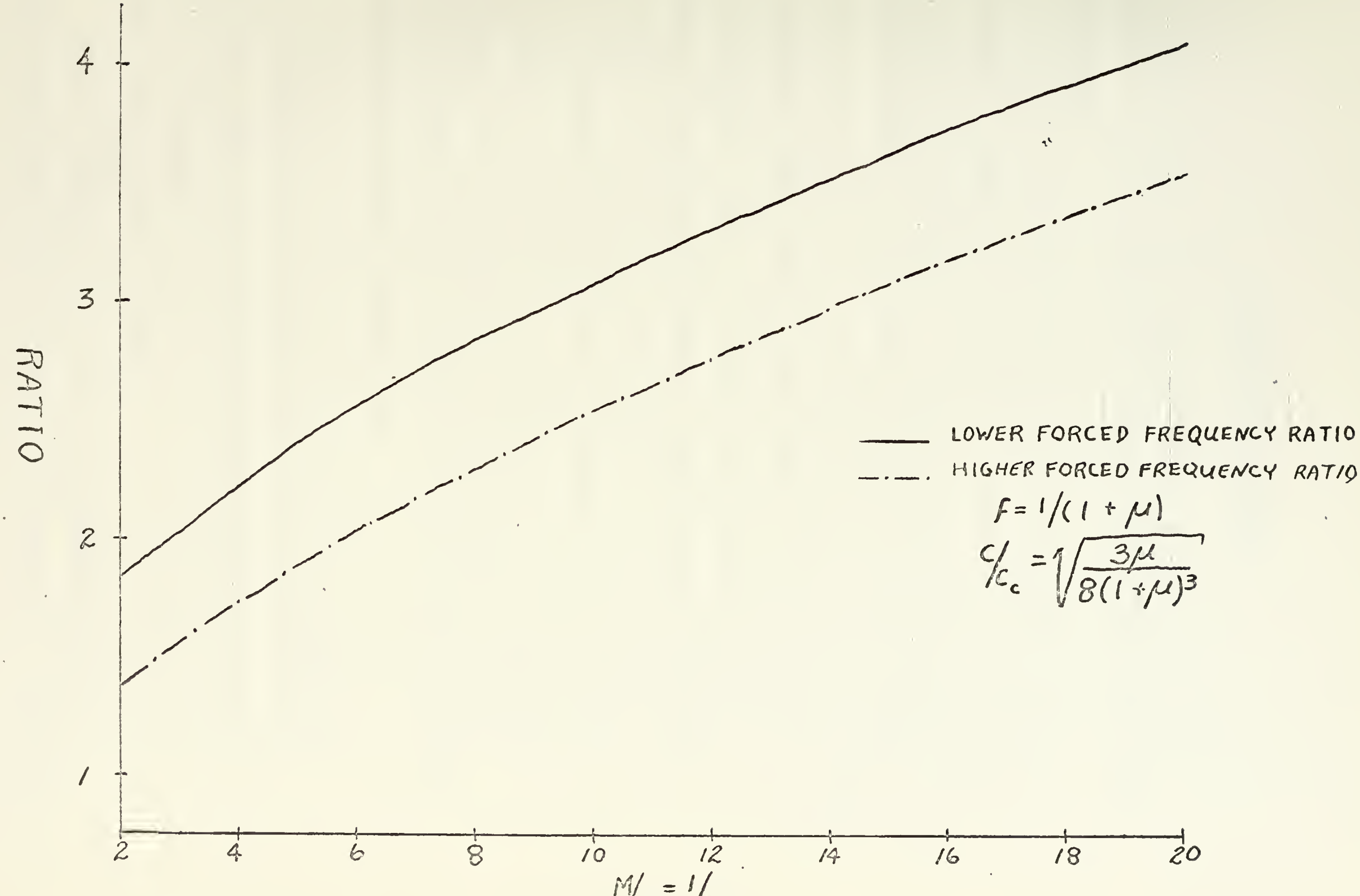

FIGURE 5. INERTIA FORCE/DAMPING FORCE VERSUS RECIPROCAL OF MASS RATIO.

DISCUSSION OF RESULTS

All derivations presented in this thesis were based on the assumption that the springs were linear and the damping coefficients were constant.

The curve describing the motion of the main mass, x_1/x_{st}, in Figure 2, although recalculated, is a repeat of the curve found on page 100 of Reference (1). It clearly describes the motion of the main mass of a two degree of freedom system with optimum tuning and optimum damping. This optimum damping used is the average value of the optimum damping found at each of the two resonant forced frequency ratios. The reason for using this average is obvious: the damping was assumed to remain constant throughout the range of operation and this average is the best compromise.

The curve of the relative motion between the main mass and the absorber mass, $(x_1-x_2)/x_{st}$, indicates that this displacement is nearly twice the displacement of the main mass. This is for the best designed vibration absorber of optimum tuning and optimum damping for a specified absorber mass.

The curve of the motion of the absorber mass, x_2/x_{st}, does not have the pronounced resonant peaks as does the motion of the main mass. This curve is not particularly significant as only relative displacements and the motion of the main system is of concern for practical applications.

Figure 3 represents the ratio of the Spring Force to the Damping Force versus the reciprocal of the mass ratio. This ratio

was calculated for the optimum tuning, the average optimum damping, and for each of the two resonant forced frequency ratios. These forced frequency ratios are derived in Appendix A. By using these parameters to calculate this force ratio and the succeeding two force ratios, the operation of the best designed vibration absorber can be analyzed at the worst operating point.

The ratio of the Spring Force to the Damping Force is essentially linear over the range going from a value of 1.5 to 4.0 for the lower forced frequency ratio and from unity to 3.5 for the higher frequency ratio.

Figure 4 represents the ratio of the Spring Force to the Inertia Force. For reasonably designed absorbers, that is with an absorber mass of 1/10 to 1/20 the mass of the main system, this ratio is essentially constant at a value of 0.95.

Figure 5 represents the ratio of the Inertia Force to the Damping Force. For the lower forced frequency ratio this force ratio goes from a value of 2.0 to 4.0 and from 1.5 to 3.5 for the higher frequency ratio.

CONCLUSIONS

1. The ratio of the Spring Force to the Damping Force varies linearly with the reciprocal of the mass ratio for specified parameters.

2. The ratio of the Spring Force to the Inertia Force is essentially constant for reasonably sized vibration absorbers.

3. The ratio of the Inertia Force to the Damping Force varies linearly with the reciprocal of the mass ratio for specified parameters.

RECOMMENDATIONS

Incorporate the data obtained in Reference (2) with the force ratios derived in this thesis to obtain curves which could be utilized in the design of visco-elastic vibration absorbers.

APPENDIX A

Determination of resonant forced frequency ratios:

On page 99 of Reference (1) a quadratic equation in g^2 is given which was derived at values for which the motion of the main mass, x_1/x_{st}, is independent of damping. This equation is:

$$(1) \quad g^4 - 2g^2 \left[\frac{1 + f^2 + \mu f^2}{2 + \mu}\right] + \frac{2f^2}{2+\mu} = 0$$

After substituting the value of optimum tuning:

$$f = \frac{1}{1+\mu}$$

equation (1) becomes

$$(2) \quad g^4 - \frac{2g^2}{\mu+1} + \frac{2}{(\mu+1)^2(\mu+2)} = 0$$

The roots of equation (2) are:

$$(3) \quad g_1^2 = \frac{1}{1+\mu}\left[1 - \sqrt{\frac{\mu}{\mu+2}}\right]$$

$$(4) \quad g_2^2 = \frac{1}{1+\mu}\left[1 + \sqrt{\frac{\mu}{\mu+2}}\right]$$

These two roots, (3) and (4), give the values of the resonant forced frequency ratios. Because in the three force ratios the forced frequency ratio is squared, (3) and (4) were substituted directly.

APPENDIX B

Summary of Data and Calculations

Forced Frequency Ratio (g)	Displacement Amplitudes		
	Main Mass (x_1)	Absorber Mass (x_2)	Relative (x_1-x_2)
0.50	1.5214	2.3008	0.8503
0.55	1.7305	2.8536	1.2623
0.60	2.0329	3.6621	1.9054
0.65	2.4482	4.7856	2.8863
0.70	2.8723	5.9704	4.1221
0.75	3.0003	6.3928	4.9983
0.80	2.8247	5.8800	5.1571
0.85	2.6655	5.1740	5.0483
0.90	2.6513	4.6411	5.0008
0.95	2.7622	4.2852	5.0656
1.00	2.9246	4.0000	5.1571
1.05	3.0122	3.6394	5.0906
1.10	2.8951	3.1074	4.6928
1.15	2.5769	2.4742	4.0170
1.20	2.1863	1.8916	3.2886
1.25	1.8270	1.4345	2.6608
1.30	1.5322	1.0990	2.1677
1.35	1.2994	0.8565	1.7911
1.40	1.1162	0.6798	1.5030

Forced Frequency Ratio (Lower)	Main Mass/Absorber Mass	Force Ratios		
		Spring/Damping	Spring/Inertia	Inertia/Damping
0.6070	2.00	1.5530	0.8407	1.8471
0.6830	3.00	1.7931	0.8733	2.0531
0.7302	4.00	2.0000	0.8944	2.2360
0.7629	5.00	2.1845	0.9092	2.4025
0.7870	6.00	2.3526	0.9203	2.5564
0.8056	7.00	2.5081	0.9288	2.7001
0.8205	8.00	2.6534	0.9357	2.8356
0.8327	9.00	2.7903	0.9413	2.9641
0.8430	10.00	2.9201	0.9460	3.0866
0.8517	11.00	3.0438	0.9500	3.2039
0.8593	12.00	3.1622	0.9534	3.3166
0.8659	13.00	3.2759	0.9564	3.4252
0.8717	14.00	3.3855	0.9590	3.5301
0.8769	15.00	3.4913	0.9613	3.6317
0.8816	16.00	3.5937	0.9633	3.7302
0.8858	17.00	3.6930	0.9652	3.8260
0.8897	18.00	3.7895	0.9669	3.9193
0.8932	19.00	3.8835	0.9684	4.0101
0.8964	20.00	3.9750	0.9697	4.0989

Forced Frequency Ratio (Higher)	Main Mass/Absorber Mass	Force Ratios		
		Spring/Damping	Spring/Inertia	Inertia/Damping
0.9822	2.00	0.9598	0.6924	1.3861
1.0165	3.00	1.2047	0.7694	1.5657
1.0327	4.00	1.4142	0.8164	1.7320
1.0414	5.00	1.6003	0.8480	1.8870
1.0463	6.00	1.7695	0.8706	2.0326
1.0492	7.00	1.9258	0.8874	2.1700
1.0509	8.00	2.0717	0.9005	2.3004
1.0518	9.00	2.0291	0.9110	2.4249
1.0523	10.00	2.3393	0.9195	2.5441
1.0525	11.00	2.4633	0.9265	2.6585
1.0524	12.00	2.5819	0.9325	2.7688
1.0522	13.00	2.6959	0.9375	2.8754
1.0519	14.00	2.8056	0.9419	2.9758
1.0516	15.00	2.9116	0.9457	3.0785
1.0511	16.00	3.0141	0.9491	3.1757
1.0507	17.00	3.1136	0.9521	3.2702
1.0502	18.00	3.2102	0.9547	3.3624
1.0498	19.00	3.3042	0.9571	3.4522
1.0493	20.00	3.3959	0.9592	3.5400

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